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General relativity and conformal invariance: I A new look at some old field equations

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Abstract. In this, the first of a pair of related papers, it is argued that general relativity allows more freedom in the choice of the metric than is usually supposed. In particular, conformally equivalent classes of metrics are to be regarded as physically equivalent. The theory is thus conformally invariant—a fact which is obscured by the non-conformally covariant way in which the field equations are usually written. Furthermore, there may be situations in which this usual way of writing the field equations is liable to give misleading results in the same way that a bad choice of coordinate system can lead to non-physical results. Another representation is given which does not suffer from this disadvantage and in a companion paper we shall apply this new representation to the problem of black holes and obtain some rather surprising results.

1. Introduction

Consider the field equations

$$R_{\iota\kappa}-\tfrac{1}{2}Rg_{\iota\kappa}=-\frac{6}{\sigma^2}T_{\iota\kappa}+\frac{4}{\sigma^2}(\sigma_{;\iota}\sigma_{;\kappa}-\tfrac{1}{4}g_{\iota\kappa}\sigma_{;l}\sigma_{;}{}^{l})+\frac{2}{\sigma}(g_{\iota\kappa}\Box^2\sigma-\sigma_{;i;\kappa}). \tag{1}$$

These are well known to be conformally invariant. O'Hanlon and Tupper (1973a, b) start from a general scalar–tensor theory and restrict it to be conformally invariant. Bramson (1974) gives a spinor formulation. Pandres and Zund (1974) derive them from Dirac's (1973) theory—which is identical in structure to the theories proposed by Omote (1971, 1974), Lord (1972), Freund (1974) and Utiyama (1975)—Hoyle and Narlikar (1964, 1974) derive them as a smooth-fluid approximation to their conformally invariant action-at-a-distance theory, and most recently Canuto *et al* (1977) have used the same equations, i.e. invariant under the transformation

$$\hat{g}_{\iota\kappa}=\Lambda^2 g_{\iota\kappa} \qquad \hat{\sigma}=\Lambda^{-1}\sigma \tag{2}$$

and to reduce in the 'conformal frame' defined by

$$\sigma=\text{constant}=\sigma_{\text{A}} \tag{3}$$

to the standard equations of general relativity with

$$G=\frac{3}{4\pi}\frac{1}{\sigma_{\text{A}}^2}. \tag{4}$$

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Now, in units with $c = \hbar = 1$ so that length and time are measured with the same unit and mass with the inverse length unit (the mass of a particle m is given by $m = q\sigma$ where q is a dimensionless constant associated with the particle, called its *inertial charge*), the transformation (2) represents a space–time-dependent change in the length unit. Thus, if we represent the length unit by a scalar field L, equation (2) corresponds to the transformation

$$\hat{L} = \Lambda^{-1} L. \tag{5}$$

The choice of length unit is purely a matter of human convention. This was recognized by Dicke (1962): 'Imagine if you will that you are told by a space traveller that a hydrogen atom on Sirius has the same diameter as one on Earth. A few moments thought will convince you that this statement is either a definition or else meaningless, and is also contained in Poincaré's (1929, 1946) geometrical conventionalism, which was explicitly endorsed by Einstein (1949).

General relativity is *par excellence* the theory in which human conventions are eliminated from the basic structure of the theory, and yet, although the choice of length unit is just as much a human convention as the choice of coordinate system, general relativity is almost always written in the restricted form with the arbitrary condition of equation (3) imposed.

The restricted form of the theory has become so ingrained into our habits of thinking that even those authors who have produced the equations (1) have failed to recognise them for what they are—the full equations of general relativity—and have regarded them as a new theory which turns out, through equation (3), to be equivalent to general relativity, which they identify with the restricted equations.

Perhaps we should give Dicke's statement (quoted earlier) more than just a few moments thought, for, once the dependence of the length unit on human conventions is realised, it becomes just as contrary to the spirit of general relativity to insist on using the same length unit in all situations as it is to insist on using the same coordinate system in all situations.

There is a danger in using the same coordinate system under all circumstances that effects which are merely due to that choice of coordinate may be taken as having some real physical significance. This is especially true in the case of singularities; think for instance of the Schwarzchild singularity.

An exactly parallel danger exists if we insist on using the same length unit in every situation. We can see this as follows.

To be an allowable length unit the scalar field must satisfy

$$0 < L < \infty \tag{6}$$

since it is obviously impossible to measure anything with a length that is itself zero or infinite. In order to preserve the condition (6) the conformal transformations must be restricted to be *non-singular*, i.e. such that

$$0 < \Lambda < \infty. \tag{7}$$

Now, suppose that we have an allowable frame, which need not necessarily be identical to the frame defined by equation (4), which we call frame A. In the allowable frame therefore, the field σ will have some specific variation over the manifold $\sigma = \sigma(x)$. In order to transform to frame A we need to use the transformation function

$$\Lambda = \sigma/\sigma_{\mathrm{A}} \tag{8}$$

for then

$$\hat{\sigma} = \Lambda^{-1}\sigma = \sigma_{\text{A}}.$$

We thus see that frame A will *only* be an allowable frame if the conformal transformation function given by equation (8) satisfies equation (7).

If we postulate that there is an allowable frame in which all particles have positive mass we can ensure that $\sigma > 0$ in this frame and hence that $\Lambda > 0$†. However, it is not so easy to dismiss the possibility that the manifold may contain singularities where σ (and hence Λ) tend to infinity.

In fact, this possibility appears very likely when we consider the wave equation for σ obtained by taking the trace of equation (1)

$$\Box^2\sigma + \tfrac{1}{6}R\sigma = T/\sigma = S \tag{9}$$

where S represents the density of inertial charge. (This can be seen as follows: represent the matter as a system of particles, with the ath particle ($a = 1, 2 \ldots$) being represented by a c^{∞} timelike curve $a^{i_{\text{A}}}(a)$ in the manifold where the parameter a represents the proper time along the curve. Let the particle have a constant inertial charge q_a and hence mass $m_a = q_a\sigma(\text{A})$ which varies as A moves along the curve. The energy–momentum tensor for the system is then

$$T^{i\kappa}(X) = \sum_a \int \delta^{(4)}(X, \text{A})[-\bar{g}(X, \text{A})]^{-1/2} m_a \frac{\mathrm{d}a^{i_{\text{A}}}}{\mathrm{d}a}\frac{\mathrm{d}a^{\kappa_{\text{A}}}}{\mathrm{d}a}\bar{g}^{i}_{i_{\text{A}}}\bar{g}^{\kappa}_{\kappa_{\text{A}}}$$

the notation being that of Hoyle and Narlikar (1964), and $\bar{g}^{i}_{i_{\text{A}}}$ being the bivector of geodetic parallel displacement. Then

$$T = \sum_a \int m_a \delta^{(4)}(X, \text{A})(-\bar{g})^{-1/2}\,\mathrm{d}a$$

so that

$$T/\sigma = S = \sum_a \int q_a \delta^{(4)}(X, \text{A})(-\bar{g})^{-1/2}\,\mathrm{d}a$$

which is the density of inertial charge.) The RHS of equation (9) is thus not explicitly dependent on σ, so that against a fixed metric background equation (9) is linear. If the sources *are* point particles, S contains δ-function terms which will lead to singularities in σ.

At this stage the argument is merely suggestive of the fact that frame A may not be allowable under certain circumstances—and hence that we should be careful about using it. However, when we actually start to solve any particular problem in the theory we need a conformal frame to work in (in the same way that we need a coordinate frame to work in), and we would like to find one which we can be sure is allowable. Such a frame could then be used as the standard against which to judge the allowability of frame A.

Fortunately, such a conformal frame is available. This is the frame B defined by Islam (1969, 1970) in his investigations of Hoyle and Narlikar's conformal theory of gravity.

† The idea of allowable and non-allowable frames has been discussed in a cosmological context by Hoyle and Narlikar (1974). They allow 'regions of negative mass' and hence discuss situations where the non-allowability is associated with $\sigma = 0$.

Frame A is defined over the whole of the manifold by the condition that σ be constant. Frame B is defined in a local region of the manifold by the condition that the *non-local* part of σ be constant. (This means that frame B is *locally adapted*, i.e. local problems can be solved in purely local terms, the distant matter only being concerned in so far as it generates the constant 'background' field.)

Thus, given a localised distribution of matter, we can divide the total mass field σ—for instance by using the Kirchoff-type integral formula derived from equation (9)—into a part generated by the local matter, $\sigma_{(l)}$ and a non-local part $\sigma_{(d)}$ which will consist of the fields generated by the non-local matter together with any totally source-free part of σ.

Within the local region we assume that, in an allowable frame, any singularities in the total mass field σ have a local cause and are not due to some 'cosmic conspiracy' involving the non-local fields combining together in some way to generate the local singularity. Thus, any singularities in σ will be in the local part $\sigma_{(l)}$ and the non-local part $\sigma_{(d)}$ will be singularity-free.

Hence frame B defined by

$$\sigma_{(d)} = \text{constant} = \sigma_0 \tag{10}$$

is an allowable frame, and is thus—while the allowability of frame A remains in doubt—to be preferred to frame A for the solution of local problems.

2. Equations in frame B

2.1. Lemma 1

The field equations in frame B, in the local region, take the form

$$\square^2\sigma = S = T/(\sigma+\sigma_0) \tag{11}$$

$$R_{i\kappa} = -\frac{6}{(\sigma+\sigma_0)^2}(T_{i\kappa} - \tfrac{1}{3}Tg_{i\kappa}) + \frac{4}{(\sigma+\sigma_0)^2}(\sigma_i\sigma_\kappa - \tfrac{1}{4}g_{i\kappa}\sigma_1\sigma^1) - \frac{2}{(\sigma+\sigma_0)}\sigma_{i\kappa}. \tag{12}$$

where from now on we use σ for $\sigma_{(l)}$ (our previous σ, the total mass field, is thus written $\sigma+\sigma_0$), and $\sigma_{;i}$ as σ_i, $\sigma_{;i;\kappa} = \sigma_{i\kappa}$.

Proof: Since $\sigma_{(d)}$ is source-free in the local region and since, as discussed in detail above, equation (9) is linear we have in the local region

$$\square^2\sigma_{(d)} + \tfrac{1}{6}R\sigma_{(d)} = 0 \tag{13}$$

and since in frame B $\sigma_{(d)}$ also satisfies equation (10) this leads to

$$R = 0 \tag{14}$$

(since all inertial charges are positive and we assume that there is some distant matter we have $\sigma_{(d)} > 0$). Using equations (14) and (10) in the field equations (1) and (9) then gives

$$R_{i\kappa} = -\frac{6}{(\sigma+\sigma_0)^2}T_{i\kappa} + \frac{4}{(\sigma+\sigma_0)^2}(\sigma_i\sigma_\kappa - \tfrac{1}{4}g_{i\kappa}\sigma_l\sigma^l) + \frac{2}{(\sigma+\sigma_0)}(g_{i\kappa}\square^2\sigma - \sigma_{i\kappa}) \tag{15}$$

and

$$\square^2\sigma = T/(\sigma+\sigma_0).$$

The second of these equations is just equation (11); substituting it into equation (15) gives the desired result (equation (12)).

In what follows we shall wish to apply these equations in the part of the local region exterior to the matter. So setting $T_{i\kappa}=0$ we get the following vacuum equations:

$$\Box^2\sigma = 0 \tag{16}$$

and

$$R_{i\kappa} = \frac{4}{(\sigma+\sigma_0)^2}(\sigma_i\sigma_\kappa - \tfrac{1}{4}g_{i\kappa}\sigma_l\sigma^l) - \frac{2}{(\sigma+\sigma_0)}\sigma_{i\kappa}. \tag{17}$$

Note the fundamental difference between this set of equations and the general field equations (1) and (9). In that case there is a redundancy, equation (9) being the trace of equation (1). This redundancy is due to the freedom of choice of conformal frame. In a specific conformal frame, such as frame A or frame B, this redundancy disappears. The situation is exactly analogous to the four extra equations one adds as coordinate conditions to eliminate the redundancy due to the freedom of choice of coordinate system.

2.2. *Lemma 2*

The Newtonian approximation to the solution of the field equations (11), (12) is

$$g_{ab} \approx -\delta_{ab} \qquad g_{a4} \approx 0 \tag{18}$$

$$g_{44} \approx 1 + \tfrac{8}{3}\phi \qquad \sigma/\sigma_0 \approx -\tfrac{1}{3}\phi \tag{19}$$

where ϕ is the Newtonian gravitational potential and $\{x^a, x^4\}$ $(a = 1, 2, 3)$ are asymptotically Cartesian coordinates.

Proof: The technique for obtaining the Newtonian approximation to the field equations is well known and can be found in any standard textbook on general relativity (see for instance Weinberg 1972). Taking the energy–momentum tensor to Newtonian order we get

$$T_{44} = T = \rho = \sum_a \sigma_0 q_a \delta(\boldsymbol{x} - \boldsymbol{x}_a) = \sum_a m_0^a \delta(\boldsymbol{x} - \boldsymbol{x}_a) \tag{20}$$

the Newtonian mass density. Then using the approximation

$$R_{44} = -\tfrac{1}{2}\nabla^2 g_{44}$$

equation (11) and the 4–4 component of equation (12) give

$$\nabla^2\, \sigma/\sigma_0 = -\rho/\sigma_0^2, \qquad \nabla^2 g_{44} = \frac{8}{\sigma_0^2}\rho.$$

Define ϕ to be such that

$$\nabla^2\phi = 3\rho/\sigma_0^2 \qquad (\phi \to 0 \text{ at infinity}) \tag{21}$$

then

$$g_{44} = 1 + \tfrac{8}{3} \qquad \sigma/\sigma_0 = -\tfrac{1}{3}\phi.$$

To show that ϕ is indeed the Newtonian potential consider the equation of motion for a particle. This is given by Hoyle and Narlikar (1964) and is

$$\frac{\mathrm{d}}{\mathrm{d}a}\left(m_a\frac{\mathrm{d}a^i}{\mathrm{d}a}\right)+m_a\Gamma^i_{\kappa 1}\frac{\mathrm{d}a^\kappa}{\mathrm{d}a}\frac{\mathrm{d}a^1}{\mathrm{d}a}=g^{i\kappa}\frac{\partial m_a}{\partial x^\kappa} \tag{22}$$

which, when approximated to Newtonian order, becomes

$$\begin{aligned}\frac{\mathrm{d}^2x^i}{\mathrm{d}t^2}&=-\frac{1}{2}\frac{\partial g_{44}}{\partial x^i}-\frac{1}{m_0^a}\frac{\partial m_a}{\partial x^i}\\&=-\frac{4}{3}\frac{\partial\phi}{\partial x^i}+\frac{1}{3}\frac{\partial\phi}{\partial x^i}\\&=-\frac{\partial\phi}{\partial x^i}\end{aligned} \tag{23}$$

which is just the Newtonian equation of motion with ϕ to be identified with the Newtonian potential.

The gravitational mass m of the system is defined by the monopole term in ϕ

$$\phi\sim -m/r \tag{24}$$

which leads to

$$m=\frac{3}{4\pi\sigma_0^2}\sum_a m_0^a=\frac{3}{4\pi\sigma_0^2}m_0 \tag{25}$$

where m_0 is the total inertial mass of the system. Thus to Newtonian order the gravitational constant G defined such that $m = Gm_0$ is given by

$$G=3/4\pi\sigma_0^2 \tag{26}$$

which agrees with equation (4) to this order.

Note that equations (18) and (19) apply only in frame B. In other conformal frames the Newtonian gravitational force is split in different proportions between metric effects and variable mass effects, neither of which is observable separately, the observable force being the combined force given by equation (23).

3. Conclusion

We have shown that general relativity needs to be conformally invariant just as much as it needs to be coordinate invariant. Furthermore, the usual way of writing the equations is possibly misleading in the same way that a coordinate singularity could be misleading. We have obtained a new representation (frame B) in which there is no such possibility. In this new representation the field equations and their Newtonian approximations have been given. Note that the conformal invariance of the theory ensures that there is no difference between the new representation and the standard one as regards the usual experimental tests of the theory.

In a subsequent paper we will show, however, that the two representations differ considerably in their predictions about black holes and singularities.

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